Note on Renewal Theory

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1 Renewal Process

Definition 1.1 (Renewal process)

Let $\{X_n, n = 1, 2, ...\}$ be a sequence of nonnegative independent random variables with a common distribution F. X_n is interpreted as the time between the $(n-1)$st and nth event. Let $N(t)$ be the number of events occured before or at time t. The counting process $\{N(t), t \geq 0\}$ is called a renewal process.

Remark

1. Note that a renewal process does not possess stationary increments and independent increments.
2. renewal process is usually using to model machine's break down, and holding time represents the time interval between two breaking down machines.

Corollary 1.1 (Strong law of large numbers for Renewal Process)

$$\mu := E[X_n] = \int_0^\infty x dF(x)$$

By the strong law of large numbers, $S_n/n \to \mu$ with probability 1 as $n \to \infty$.

Remark Hence it is impossible that $S_n \leq t$ as $n \to \infty$, so $N(t) < \infty$ for any finite t with probability 1.

Corollary 1.2 (Distribution of $N(t)$ for Renewal Process)

Letting $S_0 = 0, S_n = \sum_{i=1}^n X_i$, it follows that S_n is the time of the nth event. Here F_n denotes the distribution of $S_n = \sum_{i=1}^n X_i$.

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \qquad (N(t) = \sup\{n : S_n \leq t\}) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

Definition 1.2 (Renewal function)

$m(t) = E[N(t)]$ is called the renewal function.

$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

Remark Note that we have $m(t) < \infty$ for all $0 \le t < \infty$.

Proof

$$m(t) = E[N(t)] = \sum_{n=1}^{\infty} P\{N(t) \ge n\} = \sum_{n=1}^{\infty} P\{S_n \le t\} = \sum_{n=1}^{\infty} F_n(t) \quad \text{(Theorem ??)}$$

■

Theorem 1.1 (Strong Law of Renewal Process)

With probability 1,

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \lim_{t\to\infty} \frac{t}{N(t)} = \mu \tag{1}$$

Remark $1/\mu$ is called the rate of the renewal process

Proof When $N(t) = n, t = S_n$, we have

$$\frac{N(t)}{t} = \frac{n}{S_n} \quad \Leftrightarrow \quad \frac{t}{N(t)} = \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \to \mu$$

When $N(t) = n, t \ne S_n$, we have:, Since $t - S_n$ is less than X_{n+1}.

$$\frac{t}{N(t)} = \frac{S_n + (t - S_n)}{n} = \frac{S_n}{n} + \frac{t - S_n}{n} \to \mu$$

Or we can denote $S_{N(t)}$ as the time of the last renewal prior to or at time t, and $S_{N(t)+1}$ as the time of the first renewal after time t, then

$$S_{N(t)} \le t \le S_{N(t)+1} \quad \Rightarrow \quad \frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

- By the strong law of large numbers, $\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \to \mu$ as $t \to \infty$.
- Similarly, $\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \frac{N(t)+1}{N(t)}$

By the Squeeze Theorem, we can prove the theorem. ■

Lemma 1.1 (Central Limit Theorem for Renewal Process)

Let μ and σ^2, assumed finite, represent the mean and variance of an interarrival time. Then

$$P\left\{\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} dx \quad as \quad t \to \infty$$

Note that this theorem implies that $N(t)$ is asymptotically normally distributed with mean t/μ and variance $t\sigma^2/\mu^3$ as $t \to \infty$.

2 The Elementary Renewal Theorem

Definition 2.1 (Stopping Time)

Let $X_1, X_2...$ denote a sequence of independent random variables. An integer-valued random variable N is said to be a stopping time for the sequence $X_1, X_2, ...$ if the event $\{N = n\}$ is independent of $X_{n+1}, ...$ for all $n = 1, ...$.

Theorem 2.1 (Wald's Equation)

If $X_1, X_2, ...$ are independent and identically distributed random variables having finite expectations, and if N is a stopping time for $X_1, X_2, ...$ such that $E[N] < \infty$, then

$$E[\sum_{n=1}^{N} X_n] = E[N]E[X]$$

Example 2.1Stopping time Let $X_n, n = 1, 2...$ be independent and such that

$$P\{X_n = 0\} = P\{X_n = 1\} = \frac{1}{2}, n = 1, 2, ...$$

If we let $N = \min\{n : X_1 + ... + X_n = 10\}$, then N is a stopping time. Since by follows, $E[N] = 20$.

$$10 = E[X_1 + ... + X_N] = \frac{1}{2}E[N]$$

Example 2.2Not Stopping time Let $X_n, n = 1, 2...$ be independent and such that

$$P\{X_n = -1\} = P\{X_n = 1\} = \frac{1}{2}, n = 1, 2, ...$$

If we let $N = \min\{n : X_1 + ... + X_n = 1\}$, then N is not a stopping time. Since the follow equation is a contradiction.

$$1 = E[X_1 + ... + X_N] = 0 \cdot E[N]$$

Theorem 2.2 (The Elementary Renewal Theorem)

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

Remark This theorem is necessary, because in general, if $Z_n \to z$ with probability 1, $E[Z_n]$ may not converge to z.

Proof The proof is based on the stopping time and wald's equation. Note that the event $\{N(t) = n\}$ depends on X_{n+1}, implying that $N(t)$ is not a stopping time. Observe that

$$\begin{aligned} N(t) + 1 = n \Leftrightarrow \quad & N(t) = n - 1 \\ \Leftrightarrow \quad & X_1 + \cdots + X_{n-1} \le t, X_1 + \cdots + X_n > t \end{aligned}$$

The event $\{N(t) + 1 = n\}$ is independent of $X_{n+1}, ...$, suggesting that $N(t) + 1$ is a stopping time. From Wald's equation $m(t) < \infty$, we obtain that

$$E[X_1 + ... + X_{N(t)+1}] = E[X]E[N(t) + 1]$$

That can be rewritten as

$$E[S_{N(t)+1}] = \mu[m(t)+1]$$

Note that $S_{N(t)} \leq t < S_{N(t)+1}$, and this gives

$$\mu[m(t)+1] > t \quad \Leftrightarrow \quad m(t)+1 > t/\mu \quad \Leftrightarrow \quad m(t) > t/\mu - 1$$

This can derive the lower bound:

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \liminf_{t\to\infty} \frac{t/\mu - 1}{t} = \frac{1}{\mu}$$

When it comes to the upper bound, if $X_i \leq M$, then $S_{N(t)+1} \leq t + M$, and we have $\mu(m(t)+1) \leq t + M \to m(t) \leq (t+M)/\mu - 1$

If X_i is unbounded, we can define a new process based on $\min\{X_i, M\}$. Let $\bar{m}(t)$ be the renewal function for the new process. We have $\bar{m}(t) \geq m(t), \bar{m}(t) \leq (t+M)/\mu_M - 1$, where $\mu_M = E[min\{X_i, M\}]$. Therefore, $m(t) \leq (t+M)/\mu_M - 1$. And the upper bound is:

$$\limsup_{t\to\infty} \frac{\bar{m}(t)}{t} \leq \limsup_{t\to\infty} \frac{(t+M)/\mu_M - 1}{t} = \frac{1}{\mu_M}$$

Note that if we let $M \to \infty$, then $\mu_M \to \mu$, and the upper bound for $\frac{m(t)}{t}$ is also $\frac{1}{\mu}$, so we have

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

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3 The key Renewal Theorem

Definition 3.1 (Lattice random variable and Lattice distribution function)

A nonnegative random variable X is said to be lattice if there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} P\{X = nd\} = 1$. That is, X is lattice if it only takes on integral multiples of some nonnegative number d. The largest d is said to be the period of X.

If X is lattice and F is its distribution function, then we say F is lattice.

Theorem 3.1 (Blackwell's Theorem)

1. *If F is not lattice, then*

$$\lim_{t\to\infty} m(t+a) - m(t) = a/\mu$$

for all $a \geq 0$.

2. *If F is lattice with period d, then*

$$\lim_{n\to\infty} E[\#\text{ of renewals at } nd] = d/\mu$$

Proof

1. $\lim_{t\to\infty} m(t+a) - m(t) = \lim_{t\to\infty} (t+a)/\mu - t/\mu = a/\mu$

2. As no renewal occurs in $((n-1)d, nd)$

$$\begin{aligned}\lim_{n\to\infty} E[\,\#\text{ of renewals at } nd] &= \lim_{n\to\infty} E[N(nd) - N((n-1)d)] \\ &= \lim_{n\to\infty} m(nd) - m((n-1)d) \\ &= \lim_{n\to\infty} nd/\mu - (n-1)d/\mu = d/\mu\end{aligned}$$

If interarrivals are always positive, in the lattice case

$$P\{\text{ renewal at } nd\} = E[\,\#\text{ of renewals at } nd] \to d/\mu \quad \text{ as } n\to\infty$$

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Definition 3.2 (Direct Riemann Integrability)

Let h be a function defined on $[0\infty)$. For any $a > 0$, let $\bar{m}_n(a)$ be the supremum and $\underline{m}_n(a)$ the infinum of $h(t)$ over the interval $(n-1)a \le t \le na$. We say that h is directly Riemann integrable if $\sum_{n=1}^{\infty} \bar{m}_n(a)$ and $\sum_{n=1}^{\infty} \underline{m}_n(a)$ are finite for all $a > 0$ and

$$\lim_{a\to 0} a\sum_{n=1}^{\infty} \bar{m}_n(a) = \lim_{a\to 0} a\sum_{n=1}^{\infty} \underline{m}_n(a)$$

Theorem 3.2 (Sufficient condition for dRi)

A sufficient condition for h to be dRi is that

1. $h(t) \ge 0 \forall t \ge 0$
2. $h(t)$ *is nonincreasing*
3. $\int_0^\infty h(t)dt < \infty$

Theorem 3.3 (The Key Renewal Theorem)

If F is not lattice, and if $h(t)$ is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^\infty h(t)dt$$

Proof Note that when t is large, $m(t) \approx t/\mu$, and

$$\lim_{t\to\infty} \int_0^t h(t-x)dm(x) \approx \lim_{t\to\infty} \int_0^t h(t-x)\frac{1}{\mu}dx = \lim_{t\to\infty} \frac{1}{\mu}\int_0^t h(x)dx = \frac{1}{\mu}\int_0^\infty h(t)dt$$

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Theorem 3.4 (Blackwell vs. Key Renewal Theorem)

Blackwell's theorem and Key renewal theorem are equivalent.

Proof We prove Blackwell's theorem from the key renewal theorem. Define $h(t)$ for some $a \ge 0$. It is straightforward that $h(t)$ is dRi.

$$h(t) = \begin{cases} 1 & \text{if } 0 \le t \le a \\ 0 & \text{if } t > a \end{cases}$$

For any $t \geq a$, we have

$$\int_0^t h(t-x)dm(x) = \int_{t-a}^t dm(x) = m(t) - m(t-a)$$

Therefore,

$$\lim_{t\to\infty}[m(t+a)-m(t)] = \lim_{t\to\infty}[m(t)-m(t-a)] = \lim_{t\to\infty}\int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^\infty h(t)dt = \frac{a}{\mu}$$

■

Theorem 3.5 (Distribution of $S_{N(t)}$)

$$P\left\{S_{N(t)} \leq s\right\} = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y), \quad t \geq s \geq 0$$

Follow this, we have

$$P\left\{S_{N(t)} = 0\right\} = \bar{F}(t) \quad dF_{S_{N(t)}}(y) = \bar{F}(t-y)dm(y), 0 < y \leq t$$

Proof

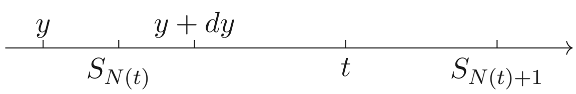
$$\begin{aligned}
P\left\{S_{N(t)} \leq s\right\} &= \sum_{n=0}^{\infty} P\left\{S_n \leq s, N(t) = n\right\} = \sum_{n=0}^{\infty} P\left\{S_n \leq s, S_{n+1} > t\right\} \\
&= \bar{F}(t) + \sum_{n=1}^{\infty} P\left\{S_n \leq s, S_{n+1} > t\right\} \\
&= \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^\infty P\left\{S_n \leq s, S_{n+1} > t \mid S_n = y\right\} dF_n(y) \\
&= \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^s \bar{F}(t-y)dF_n(y) && (P\{S_{n+1} > t | S_n = y\} = \bar{F}(t-y)) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t-y)d\left(\sum_{n=1}^{\infty} F_n(y)\right) \\
&= \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y) && (\text{Definition } 1.2)
\end{aligned}$$

The proof for the $P\left\{S_{N(t)} = 0\right\} = \bar{F}(t)$ is simple, since $S_{N(t)} = 0$ means $N(t) = 0$. As for the proof of the later, note that

$$\begin{aligned}
dm(y) &\approx m(y+dy) - m(y) = E[\#\text{ renewals in } (y, y+dy)] \\
&\approx P\{\text{ renewal occurs in } (y, y+dy)\}
\end{aligned}$$

The second approximation is obtained because there is at most one renewal in $(y, y+dy)$ for small dy with a very high probability. So

$$\begin{aligned}
dF_{S_{N(t)}}(y) &= P\left\{S_{N(t)} \in (y, y+dy)\right\} \\
&= P\left\{\begin{array}{c}\text{renewal occurs in } (y, y+dy), \\ \text{next interarrival } > t-y\end{array}\right\} \quad (\text{Figure } 3) \\
&= dm(y)\bar{F}(t-y)
\end{aligned}$$



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4 Alternating Renewal Process

Definition 4.1 (Alternating Renewal Process)

Consider a system that can be in one of two states: on or off. Initially it is on and it remains on for a time Z_1, it then goes off and remains off for a time Y_1, it then goes on for a time Z_2, then off for a time Y_2; then on, and so forth. Suppose the two sequences $\{Z_n\}$ and $\{Y_n\}$ are i.i.d, and they may be dependent. In other words, each time the process goes on everything starts over again, but when it goes off we allow the length of the off time to depend on the previous time. Let H be the distribution of Z_n, G the distribution of Y_n, and F the distribution of $Z_n + Y_n$. Furthermore, let

$$P(t) = P\{\textit{system is on at time } t\}$$

Theorem 4.1 (Lim P(t) in alternating renewal process)

If $E[Z_n + Y_n] < \infty$, and F is nonlattice, then

$$\lim_{t\to\infty} P(t) = \frac{E[Z_n]}{E[Z_n] + E[Y_n]}$$

Proof Say that a renewal takes place each time the system goes on.

$$\begin{aligned}P(t) =& E\left[P\left\{\text{ on at } t \mid S_{N(t)}\right\}\right]\\ =& P\left\{\text{ on at } t \mid S_{N(t)} = 0\right\} P\left\{S_{N(t)} = 0\right\}\\ &+ \int_0^\infty P\left\{\text{ on at } t \mid S_{N(t)} = y\right\} dF_{S_{N(t)}}(y) \quad \text{since } dF_{S_{N(t)}}(y) : 0 < y \le t\\ =& P\left\{\text{ on at } t \mid S_{N(t)} = 0\right\} \bar{F}(t)\\ &+ \int_0^t P\left\{\text{ on at } t \mid S_{N(t)} = y\right\} \bar{F}(t-y) dm(y) \quad \text{Theorem 3.5}\end{aligned}$$

Note that $S_{N(t)} = 0 \Leftrightarrow Z_1 + Y_1 > t$ and given that $S_{N(t)} = 0$, on at t $\Leftrightarrow Z_1 > t$:

$$P\left\{\text{ on at } t \mid S_{N(t)} = 0\right\} = P\left\{Z_1 > t \mid Z_1 + Y_1 > t\right\} = \frac{\bar{H}(t)}{\bar{F}(t)}$$

Suppose that $N(t) = n$, we have $S_{N(t)} = y \Leftrightarrow Z_{n+1} + Y_{n+1} > t - y$, and given that $S_{N(t)} = y$, on at t $\Leftrightarrow Z_{n+1} > t - y$:

$$\begin{aligned}P\left\{\text{ on at } t \mid S_{N(t)} = y\right\} &= P\{Z > t - y \mid Z + Y > t - y\}\\ &= \frac{\bar{H}(t-y)}{\bar{F}(t-y)}\end{aligned}$$

Or we can derive it another way:

$$\begin{aligned}&P\left\{\text{ on at } t \mid S_{N(t)} = y\right\}\\ =&\sum_n P\left\{\text{ on at } t \mid S_{N(t)} = y, N(t) = n\right\} P\{N(t) = n\}\end{aligned}$$

Conditioning on $S_{N(t)} = y$ and $N(t) = n$, on at t $\Leftrightarrow Z_{n+1} > t - y$. The second part

$S_{N(t)} = y, N(t) = n \Leftrightarrow S_n = y, S_n \le t, S_{n+1} > t \Leftrightarrow \sum_{i=1}^{n}(Z_i + Y_i) = y, Z_{n+1} + Y_{n+1} > t - y$.

$$\begin{aligned}&P\left\{\text{ on at } t \mid S_{N(t)} = y, N(t) = n\right\}\\ =&P\left\{Z_{n+1} > t - y \mid \sum_{i=1}^{n}(Z_i + Y_i) = y, Z_{n+1} + Y_{n+1} > t - y\right\}\\ =&P\left\{Z_{n+1} > t - y \mid Z_{n+1} + Y_{n+1} > t - y\right\}\\ =&\frac{\bar{H}(t-y)}{\bar{F}(t-y)}\end{aligned}$$

Hence, we have

$$\begin{aligned}&P\left\{\text{ on at } t \mid S_{N(t)} = y\right\}\\ =&\sum_n P\left\{\text{ on at } t \mid S_{N(t)} = y, N(t) = n\right\} P\{N(t) = n\}\\ =&\sum_n \frac{\bar{H}(t-y)}{\bar{F}(t-y)} P\{N(t) = n\}\\ =&\frac{\bar{H}(t-y)}{\bar{F}(t-y)}\left(\sum_n P\{N(t) = n\}\right)\\ =&\frac{\bar{H}(t-y)}{\bar{F}(t-y)}\end{aligned}$$

Return to the calculation of $P(t)$, we have

$$\begin{aligned}P(t) =&P\left\{\text{ on at } t \mid S_{N(t)} = 0\right\}\bar{F}(t)\\ &+ \int_0^t P\left\{\text{ on at } t \mid S_{N(t)} = y\right\}\bar{F}(t-y)dm(y)\\ =&\bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y)\end{aligned}$$

As $\bar{H}(t) \to 0$ as $t \to \infty$, by the key renewal theorem, we have

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\int_0^t \bar{H}(t-y)dm(y) = \frac{1}{\mu_F}\int_0^\infty \bar{H}(t)dt = \frac{E\left[Z_n\right]}{E\left[Z_n\right] + E\left[Y_n\right]}$$

Similarly, if we let $Q(t) = P\{\text{off at } t\} = 1 - P(t)$, then $Q(t) \to \frac{E[Y_n]}{E[Z_n]+E[Y_n]}$. In addition, the fact the system was initially on makes no difference in the limit. ■

Lemma 4.1 (Multiple states for alternating renewal process (Song, 2020, PS. 2))

A process is in one of n states, $1, 2, ..., n$. Initially it is in state 1, where it remains for an amount of time having distribution F_1. After leaving state 1 it goes to state 2, where it remains for a time having distribution F_2. When it leaves 2 it goes to state 3, and so on. From state n it returns to 1 and starts over. Then

$$\lim_{t\to\infty} P\{\text{ process is in state } i \text{ at time } t\} = \frac{\int_0^\infty x dF_i(x)}{\sum_{j=1}^{n}\int_0^\infty x dF_j(x)}$$

Proof On the basis of alternating renewal process, we can calculate the prob. of state $1, ..., i$. Then we conduct successive difference backwards. ■

Theorem 4.2 (Excess Life and Agev (Song, 2020, PS. 2))

Consider a renewal process and let $Y(t)$ denote the time from t until the next renewal and let $A(t)$ be the time from t since the last renewal. $Y(t)$ is called the excess or residual life at t, and $A(t)$ is called the age at t.

$$Y(t) = S_{N(t)+1} - t \quad \text{and} \quad A(t) = t - S_{N(t)}$$

If the interarrival distribution is nonlattice and $\mu < \infty$, then

$$\lim_{t\to\infty} P\{Y(t) \le x\} = \lim_{t\to\infty} P\{A(t) \le x\} = \int_0^x \bar{F}(y)dy/\mu$$

$$\lim_{t\to\infty} E[A(t)] = \lim_{t\to\infty} E[Y(t)] = \frac{E\left[X_1^2\right]}{2E\left[X_1\right]}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t Y(s)ds = \frac{E\left[X_1^2\right]}{2E\left[X_1\right]}$$

Remark

1. $A(t) \ge x \Leftrightarrow 0$ events in the interval $(t-x, t]$
2. $Y(t) > x \Leftrightarrow 0$ events in the interval $(t, t+x]$
3. $P\{Y(t) > x\} = P\{A(t+x) \ge x\}$

Proof To derive $P\{A(t) \le x\}$, let an on-off cycle correspond to a renewal and say that the system is "on" at time t if the age at t is less than or equal to x. Note that x is given, and the length between every renewal is varied, when the length is smaller than x, then the system is always "on" in this interval, when the length is larger than x, then the system is "on" in the first x interval and "off" in the remaining interval, just as the figure 1.

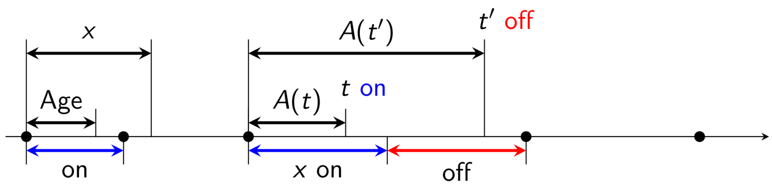

Figure 1: Excess Life and Age

Since $A(t) \le x \leftrightarrow$ on at$t$, and let $Y_n = min\{X_n, x\}$, from the alternating renewal process, we have

$$\lim_{t\to\infty} P\{A(t) \le x\} = \lim_{t\to\infty} P\{\text{ on at } t\} = \frac{E[\min(X,x)]}{E[X]}$$
$$= \int_0^\infty P\{\min(X,x) > y\}dy/E[X]$$

$$P\{\min(X,x) > y\} = P\{X > y, x > y\} = \begin{cases} 0 & \text{if } y \ge x \\ P\{X > y\} = \bar{F}(y) & \text{if } y < x \end{cases}$$

$$\lim_{t\to\infty} P\{A(t) \le x\} = \int_0^x \bar{F}(y)dy/\mu$$

Similarly, we say that the system is "off" at time t if the excess life at t is less then or equal to x and "on" otherwise. Thus the off time in a cycle is $min(X, x)$, and so

$$\lim_{t\to\infty} P\{Y(t) \le x\} = \lim_{t\to\infty} P\{\text{ off at } t\} = \frac{E[\min(X,x)]}{E[X]} = \int_0^x \bar{F}(y)dy/\mu$$

■

Definition 4.2 (Inspection Paradox)

We denote $X_{N(t)+1} = S_{N(t)+1} - S_{N(t)} = A(t) + Y(t)$ as the length of renewal interval that contains the point t, however, $X_{N(t)+1}$ do not have the same distribution as X_n, as the figure 2 shows.

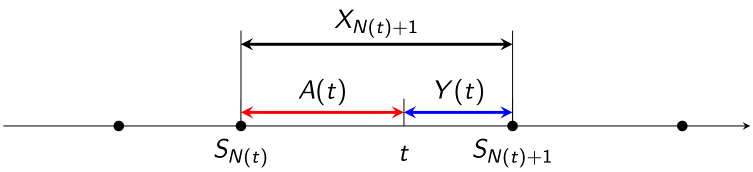

Figure 2: $X_{N(t)+1}$

Remark That is, compared with an ordinary renewal interval, the interval containing the point t is more likely to have a length greater than x. The explanation is simple, the renewal process contains many (infinite) renewal intervals, and it is more likely that a larger interval will cover the point t. Therefore, it is plausible that an interval covering the point t should be "stochastically" longer than an ordinary interval.

Proof

$$P\left\{X_{N(t)+1} > x\right\} = E\left[P\left\{X_{N(t)+1} > x \mid S_{N(t)}\right\}\right]$$

For all $s \in [0, t]$, consider $P\left\{X_{N(t)+1} > x \mid S_{N(t)} = s\right\}$. Suppose that $N(t) = n$, then $S_{N(t)} = s \leftrightarrow X_{n+1} > t - s$, and $X_{N(t)+1} > x \leftrightarrow X_{n+1} > x$, so

$$P\left\{X_{N(t)+1} > x \mid S_{N(t)} = s\right\} = P\{X > x \mid X > t - s\} = \frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)}$$

Another computation: Conditioning on $S_{N(t)} = s$ and $N(t) = n$, $X_{N(t)+1} > x \leftrightarrow X_{n+1} > x$. $S_{N(t)} = s, N(t) = n \leftrightarrow S_n = s, S_n \le t, S_{n+1} > t \leftrightarrow \sum_{i=1}^n X_i = s, X_{n+1} > t - s$.

$$\begin{aligned} &P\left\{X_{N(t)+1} > x \mid S_{N(t)} = s\right\} \\ =&\sum_n P\left\{X_{N(t)+1} > x \mid S_{N(t)} = s, N(t) = n\right\} P\{N(t) = n\} \end{aligned}$$

$$\begin{aligned} &P\left\{X_{N(t)+1} > x \mid S_{N(t)} = s, N(t) = n\right\} \\ =&P\left\{X_{n+1} > x \mid \sum_{i=1}^n X_i = s, X_{n+1} > t - s\right\} \\ =&P\left\{X_{n+1} > x \mid X_{n+1} > t - s\right\} \\ =&\frac{P\left\{X_{n+1} > \max\{x, t-s\}\right\}}{P\left\{X_{n+1} > t - s\right\}} \\ =&\frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)} \end{aligned}$$

$$\begin{aligned}
&P\left\{X_{N(t)+1}>x \mid S_{N(t)}=s\right\} \\
=&\sum_n P\left\{X_{N(t)+1}>x \mid S_{N(t)}=s, N(t)=n\right\} P\{N(t)=n\} \\
=&\sum_n \frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)} P\{N(t)=n\} \\
=&\frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)}\left(\sum_n P\{N(t)=n\}\right) \quad \text{Independent of } n \\
=&\frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)}
\end{aligned}$$

Based on this result, we have

$$\frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)}=\begin{cases}\bar{F}(t-s)/\bar{F}(t-s)=1 \geq \bar{F}(x) & \text{if } x<t-s \\ \bar{F}(x)/\bar{F}(t-s) \geq \bar{F}(x) & \text{if } x \geq t-s\end{cases}$$

$$P\left\{X_{N(t)+1}>x \mid S_{N(t)}=s\right\}=\frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)} \geq \bar{F}(x)$$

$$P\left\{X_{N(t)+1}>x \mid S_{N(t)}\right\} \geq \bar{F}(x)$$

$$P\left\{X_{N(t)+1}>x\right\}=E\left[P\left\{X_{N(t)+1}>x \mid S_{N(t)}\right\}\right] \geq \bar{F}(x)$$

■

Proof [Another proof based on alternating renewal process] Let an on-off cycle correspond to a renewal interval, and say that the system is "on" at time t if $X_{N(t)+1}>x$, that is, the system is either totally on during a cycle (if the renewal interval is greater than x) or totally off otherwise. Thus we have $P\{X_{N(t)+1}>x\}=P\{\text{on at time}t\}$. And by the theorem of alternating renewal process, we have

$$\begin{aligned}
\lim_{t \rightarrow \infty} P\left\{X_{N(t)+1}>x\right\} &=\frac{E[\text{ on time in cycle }]}{\mu} \\
&=\frac{E[E[\text{ on time in cycle } \mid \text{ cycle length }]]}{\mu} \\
&=\frac{\int_0^{\infty} E[\text{ on time in cycle } \mid \text{ cycle length } =y] dF(y)}{\mu} \\
&=\int_x^{\infty} y dF(y)/\mu \quad \text{When cycle<}x\text{, off; elif cycle>x, the on time=}y
\end{aligned}$$

$$\begin{aligned}
\lim_{t \rightarrow \infty} P\left\{X_{N(t)+1} \leq x\right\} &=1-\lim_{t \rightarrow \infty} P\left\{X_{N(t)+1}>x\right\} \\
&=1-\int_x^{\infty} y dF(y)/\mu \\
&=\frac{1}{\mu}\left(\int_0^{\infty} y dF(y)-\int_x^{\infty} y dF(y)\right) \\
&=\int_0^x y dF(y)/\mu
\end{aligned}$$

As $t \rightarrow \infty$, we have $P\{\text{an interval is of length } (y, y+dy) \text{ and contains } t\} \approx y dF(y)/\mu$. Note that this probability is also equivalent to the product of the conditional probability and the probability of an interval is the length of $(y, y+dy)$ (which is $dF(y)$), so the conditional probability $P\{\text{an interval contains } t | \text{it is of length } (y, y+dy)\} \approx y/\mu$. That is, in the limit (as $t \to \infty$), an interval of length y is y times more likely to cover t than one of length 1. As a result, an interval covering t should be "stochastically" longer than an ordinary interval. ■

Lemma 4.2 ($\lim_{t\to\infty} P\{X_{N(t)+1} \le x\}$)

5 Delayed Renewal Process

Definition 5.1 (Delayed Renewal Process)

Let $\{X_n, n = 1, 2, ...\}$ be a sequence of independent nonnegative random variables with X_1 having distribution G, and X_n having distribution F, $n > 1$. Let $S_0 = 0, S_n = \sum_{i=1}^{n} X_i, n \ge 1$, and define

$$N_D(t) = sup\{n : S_n \le t\}$$

The stochastic process $\{N_D(t), t \ge 0\}$ is called a general or a delayed renewal process.

6 Renewal Reward Process

Definition 6.1 (Renewal Reward Process)

Consider a renewal process $\{N(t), t \ge 0\}$ having interarrival times $X_n, n \ge 1$ with distribution F, and suppose that at the time of the nth renewal we receive a reward R_n. Assume that the pairs $(X_n, R_n), n \ge 1$, are independent and identically distributed. Note that R_n are i.i.d, and R_n may depend on X_n. Let

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

which represents the total reward earned by time t. Let

$$E[R] = E[R_n] \quad E[X] = E[X_n]$$

Theorem 6.1 (Theorem for Renewal Reward Process)

If $E[R] < \infty, E[X] < \infty$, then

- *with probability 1, $\frac{R(t)}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$*
- *$\frac{E[R(t)]}{t} \to \frac{E[R]}{E[X]}$ as $t \to \infty$*

Remark If we say that a cycle is completed every time a renewal occurs, then the theorem states that the expected long-run average return is just the expected return earned during a cycle, divided by the expected time of a cycle. The first point is a generalization of the strong law for renewal processes, and the second point is a generalization of the elementary renewal theorem.

This theorem remains true if the reward is earned gradually during the renewal cycle. If we assume that the reward accumulates at a random rate $r(t)$ for any $t \ge 0$, then the total reward earned by time t is represented by $R(t) = \int_{s=0}^{t} r(s)ds$. And the theorem holds if we let R denote the reward earned in a cycle, i.e., $R = \int_{s=0}^{X_1} r(s)ds$.

Proof

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \left(\frac{\sum_{n=1}^{N(t)} R_n}{N(t)}\right)\left(\frac{N(t)}{t}\right)$$

Note that the first part converges to $E[R]$ as $t \to \infty$ by the strong law of large numbers, and the later converges to $\frac{1}{E[X]}$ as $t \to \infty$ by the strong law for renewal processes. ■

Theorem 6.2 (The Elementary Renewal Theorem for Renewal Reward Process (Song, 2020, PS. 2))

Assume that F is not lattice, $P\{R_1 \geq 0\} = 1$ and $E[X_1 R_1] < \infty$.

$$\lim_{t\to\infty} \frac{E[R(t)]}{t} \to \frac{E[R_1]}{E[X_1]}$$

Proof Firstly we have $E[R(t)] = (m(t)+1)E[R_1] - E\left[R_{N(t)+1}\right]$ by a stopping time $N(t)+1$. By assumption, we have $\lim_{t\to\infty} E\left[R_{N(t)+1}\right] = \frac{E[X_1 R_1]}{E[X_1]}$. Combine them we can prove it.

$$E\left[R_{N(t)+1}\right] = E\left[R_{N(t)+1} \mid S_{N(t)} = 0\right]\bar{F}(t) + \int_0^t E\left[R_{N(t)+1} \mid S_{N(t)} = s\right]\bar{F}(t-s)dm(s)$$
$$= E\left[R_1 \mid X_1 > t\right]\bar{F}(t) + \int_0^t E\left[R_1 \mid X_1 > t-s\right]\bar{F}(t-s)dm(s)$$

■

Theorem 6.3 (Blackwell's Theorem for Renewal Reward Process (Song, 2020, PS. 2))

Assume that F is not lattice, $P\{R_1 \geq 0\} = 1$ and $E[X_1 R_1] < \infty$.

$$\lim_{t\to\infty} E[R(t+a) - R(t)] \to a\frac{E[R_1]}{E[X_1]}$$

Proof Based on the former proof. ■

Example 6.1Car's Life Car's life is a random variable with distribution F. An individual has a policy of trading in his car either when it fails or reaches the age of A. Let $R(A)$ denote the resale value of an A-year-old car. There is no resale value of a failed car. Let C_1 denote the cost of a new car and suppose that an additional cost C_2 is incurred whenever the car fails.

1. Say that a cycle begins each time a new car is purchased. The long-run average cost per unit time is $\frac{C_1 + C_2 F(A) - R(A)\bar{F}(A)}{\int_0^A x dF(x) + A\bar{F}(A)}$.
2. Say that a cycle begins each time a car in use fails. The long-run average cost per unit time is $\frac{C_1 + C_2 F(A) - R(A)\bar{F}(A)}{\int_0^A x dF(x) + A\bar{F}(A)}$.

Solution *The first case: simple, easy to see the expected length of a cycle is*

$$E[\min\{X, A\}] = \int_0^A x dF(x) + A\bar{F}(A)$$

The expected cost of a cycle is

$$(C_1 + C_2) P(X \leq A) + (C_1 - R(A)) P(X > A) = C_1 + C_2 F(A) - R(A)\bar{F}(A)$$

The second case: note that there may be some cars (N) not fail in the cycle, and the number *follows $G(F(A))$. However, it is easier to see that N is a stopping time.*

$$E[\text{ cost of a cycle }] = E[N]E[\text{ cost to use a car }] = E[N]\left(C_1 + C_2F(A) - R(A)\bar{F}(A)\right)$$

$$E[\text{ time of a cycle }] = E[N]E[\text{ time to use a car }] = E[N]\left(\int_0^A xdF(x) + A\bar{F}(A)\right)$$

Bibliography

Song, Miao (2020). *LGT6202 Stochastic Models and Decision under Uncertainty*.